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AN APPROXIMATE SOLUTION FOR AXIALLY SYMMETRIC FLOW OVER A CONE WITH AN ATTACHED SHOCK WAVE

By Richard A. Hord

Langley Aeronautical Laboratory
Langley Field, Va.

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SUMMARY

It is shown that the streamlines in an angular neighborhood of the surface of an unyawed circular cone with an attached shock wave are, to a first approximation, portions of hyperbolas. This fact is used as a basis for the development of an approximate solution in which the shock-wave orientation and the flow field behind the shock wave are given explicitly in terms of the free-stream Mach number, the vertex angle of the body cone, and the ratio of specific heats of the gas. The approximate solution is compared with other approximate solutions for the cone.

INTRODUCTION

It is well known that within calculable limits the adiabatic steady flow of an inviscid perfect gas over a nonlifting circular cone assumes a relatively simple form. The essential characteristic of this type of flow is an attached cone-shaped shock wave behind which the velocity and state of the gas are constant on conical surfaces whose axes and vertices coincide with those of the body cone and the shock-wave cone.

The basic features of this type of supersonic flow, together with a graphical method for calculating the flow field by starting with a conical shock wave of given vertex angle and entering air at a given Mach number, were described by Busemann (ref. 1). Taylor and Maccoll (ref. 2) conducted a comprehensive investigation of the problem in which exact equations governing the flow were developed and solved numerically. The numerical results were shown to bear excellent agreement with the experimentally determined values. Subsequently, Maccoll (ref. 3) extended the numerical computations to cones with larger vertex angles by using solutions in series for the velocity components in the flow field behind the shock wave and calculated a maximum possible value for the vertex angle of a cone with an attached shock wave. The most extensive tables and graphs of the flow variables, which were prepared on the basis of the work of Taylor and Maccoll, are contained in reference 4.

Even though axially symmetric flow over a cone with an attached shock wave is one of the few compressible-flow problems which admit of precise solutions with reasonable computational labor, approximate solutions in closed form are of use in certain cases. Approximate solutions for the cone may be used in the rapid calculation of supersonic flow over pointed bodies of revolution. As a second example, an approximate solution relating the velocity field behind the shock wave to the velocity at the body-cone surface may, in some instances, be combined with information on strong shock waves in a real gas to obtain estimates of the effects of vibrational excitation and dissociation on the flow over conical tips in the hypersonic range.

The first approximate solution for the cone resulted from the linearized treatment by Von Kármán and Moore (ref. 5) of the axially symmetric supersonic flow over bodies of revolution. In addition to the further development of linearized theory (for example, ref. 6 by Lighthill), Van Dyke (ref. 7) has developed a second-order theory in which a first-order solution of a given problem constitutes the starting point for an iteration procedure aimed at obtaining closer approximations. As one might expect, the increase in accuracy is accompanied by a considerable increase in computational effort.

Since the linear and second-order theories are based upon the assumption of small perturbations and flow that is free of shock waves, they are definitely limited, particularly in the realm of high Mach numbers, both in accuracy and in range of applicability.

The conical-shock-expansion method of Eggers and Savin (ref. 8) for calculating the flow over pointed bodies of revolution at high supersonic airspeeds utilizes approximate solutions for the cone to determine the nature of the flow in the neighborhood of the vertex. Since the existence of an attached shock wave is essential to these approximations, they are not subject to the same limitations as the perturbation methods and, in fact, achieve their best accuracy in the high Mach number range. Of the three approximate solutions presented for the cone, only the third (ref. 8, appendix A) can be applied with ease comparable to that of the small-perturbation solution of Von Kármán and Moore.

In the present report, it is shown that in an angular neighborhood of the body cone the streamlines are, to a first approximation, portions of hyperbolas. In keeping with the conical nature of the flow field behind the shock wave, the hyperbolas of this family are magnifications of one another with respect to their common center, the cone vertex. The generators of the body cone (the whole lines, rather than the half-lines which terminate at the vertex) are the asymptotes of the hyperbolas of the family. Thus, the conjugate axes of the hyperbolas all lie on the cone axis, and the curves themselves are symmetric about the plane normal to the cone axis at the vertex.

In the development of the present approximate solution for the nonlifting cone with an attached shock wave, the streamlines behind the shock wave are assumed to be hyperbolas of the form just described. Without further approximation being necessary, it is found that an approximate solution can be derived which yields expressions for the shock-wave angle, the pressure ratio across the shock wave, and the surface velocity in terms of the free-stream Mach number, the cone angle, and the ratio of specific heats.

SYMBOLS

a	local speed of sound, referred to maximum velocity c
c	maximum or limiting velocity which would be attained by expanding adiabatically into a vacuum
C_p	pressure coefficient
M	free-stream Mach number
p	static pressure
u	radial velocity component (as in polar-coordinate system), positive outward along ray from cone vertex, referred to maximum velocity c (fig. 1)
v	normal velocity component (as in polar-coordinate system), positive with increasing θ, referred to maximum velocity c (fig. 1)
v_r	radial velocity component normal to cone axis (as in axially symmetric cylindrical coordinate system), positive outward from cone axis, referred to maximum velocity c (fig. 1)
v_x	velocity component parallel to cone axis, positive in positive direction of free stream, referred to maximum velocity c (fig. 1)
x,r	axially symmetric cylindrical coordinates with origin at cone vertex and x-axis coincident with cone axis (fig. 1)
γ	ratio of specific heat at constant pressure to specific heat at constant volume
$\epsilon_3,\bar{\epsilon}_3$	third-order remainder terms, as described in text

θ angle which ray emanating from cone vertex makes with cone axis (fig. 1)

$\tau \equiv \tan\theta$

Subscripts:

s refers to conditions at body-cone surface

w refers to conditions immediately behind shock wave

1 refers to free-stream conditions (not used in case of free-stream Mach number)

APPROXIMATION TO THE FORM OF THE STREAMLINES NEAR THE SURFACE

The second-order ordinary differential equation which Taylor and Maccoll derived as governing the conical flow field about a nonlifting circular cone with an attached shock wave is equivalent to the following system of first-order differential equations (see, for example, ref. 4, pp. vii and viii, or ref. 2, pp. 281 and 282):

$$\left.\begin{aligned}\frac{du}{d\theta} &= v\\ \frac{dv}{d\theta} &= \frac{uv^2 - a^2(2u + v\cot\theta)}{a^2 - v^2}\end{aligned}\right\}\tag{1}$$

where the local speed of sound a is related to the radial and normal velocity components u and v (fig. 1) by

$$a^2 = \frac{\gamma - 1}{2}\left(1 - u^2 - v^2\right)\tag{2}$$

Here, for the sake of simplicity, the unit of velocity has been taken as the maximum velocity c which would be attained by allowing the gas to flow adiabatically into a vacuum.

For the purpose of the present analysis, it is convenient to transform the system of equations (1). Let v_x be the velocity component parallel to the axis of the cone (the line whose equation is $\theta = 0$) and positive in the positive direction of the free stream. Moreover, let v_r denote the radial velocity component perpendicular to the cone axis and positive outward from this axis (fig. 1). Since the flow is conical, v_x and v_r are functions of θ. The dependent variables v_x and v_r are related to the dependent variables u and v by the transformation

$$u = v_x \cos\theta + v_r \sin\theta \tag{3a}$$

$$v = -v_x \sin\theta + v_r \cos\theta \tag{3b}$$

whose inverse is

$$v_x = u \cos\theta - v \sin\theta \tag{4a}$$

$$v_r = u \sin\theta + v \cos\theta \tag{4b}$$

With the change of dependent variables defined by equations (3), the system of differential equations (1) takes the form

$$\frac{dv_x}{d\theta} = -\tan\theta \frac{dv_r}{d\theta} = \frac{a^2 v_r}{a^2 - \left(v_r \cos\theta - v_x \sin\theta\right)^2} \tag{5}$$

and equation (2) becomes

$$a^2 = \frac{\gamma - 1}{2}\left(1 - {v_x}^2 - {v_r}^2\right) \tag{6}$$

Dividing the system of equations (5) by $1 + \tan^2\theta$ and noting that

$$\left(1 + \tan^2\theta\right)d\theta = \sec^2\theta \; d\theta = d(\tan\theta)$$

gives, in conical coordinates,

$$\frac{dv_x}{d\tau} = -\tau \frac{dv_r}{d\tau} = \frac{a^2 v_r}{a^2\left(1 + \tau^2\right) - \left(v_r - \tau v_x\right)^2} \tag{7}$$

where $\tau \equiv \tan\theta$. (The fact that this form of the equations has also been given in ref. 9, p. 356, was recently noted.) The system of differential equations (7) is, then, equivalent to the second-order differential equation derived by Taylor and Maccoll.

Even though the system of equations (7) apparently does not admit of a solution in closed form, certain interesting and useful facts about the existence and nature of a solution are obtainable from the theory of functions of real variables. Consideration is restricted to the cases of physical interest, that is, $0 < \tau_s < \infty$, $1 < \gamma < \infty$, and $0 < u_s < 1$.

It follows from the theory of functions of real variables that a unique solution of equations (7) exists in a neighborhood of $\tau = \tau_s$ such that the functions v_x and v_r and their first derivatives with respect to τ are continuous functions. (See, for example, ref. 10, p. 357.) Therefore, by successively differentiating equations (7), it is evident that the higher derivatives are also continuous in a neighborhood of τ_s. These results imply that the velocity ratio v_r/v_x and its derivatives with respect to $\cot\theta = 1/\tau$ must be continuous in a neighborhood of τ_s. Consequently, it is possible to apply Taylor's theorem with a remainder (ref. 10, p. 105) to the development of the function v_r/v_x in powers of $\cot\theta - \cot\theta_s = \frac{1}{\tau} - \frac{1}{\tau_s}$. The evaluation of the coefficients for this development is a matter of straightforward calculation utilizing equations (7) together with the boundary condition

$$\frac{v_{r,s}}{v_{x,s}} = \tau_s \tag{8}$$

which is equivalent to the physical statement that the flow at the cone surface is along a generator. In particular, the term of zero order in the Taylor development is given directly by equation (8). The coefficient of the first-order term may be determined as follows:

$$\begin{aligned}
\left[\frac{d}{d(1/\tau)}\frac{v_r}{v_x}\right]_s &= \left(-\tau^2\frac{d}{d\tau}\frac{v_r}{v_x}\right)_s \\
&= \left[-\tau^2\left(\frac{v_r}{v_x}\frac{1}{v_r}\frac{dv_r}{d\tau} - \frac{v_r^2}{v_x^2}\frac{1}{v_r}\frac{dv_x}{d\tau}\right)\right]_s \\
&= \tau_s^2
\end{aligned}$$

Similarly, the second-order coefficient is found, after some calculation, to be

$$\frac{1}{2!}\left[\frac{d^2}{d(1/\tau)^2}\frac{v_r}{v_x}\right]_s = \frac{\tau_s{}^5}{1 + \tau_s{}^2}$$

The Taylor development, therefore, has the form

$$\frac{v_r}{v_x} = \tau_s + \tau_s{}^2\left(\frac{1}{\tau} - \frac{1}{\tau_s}\right) + \frac{\tau_s{}^5}{1 + \tau_s{}^2}\left(\frac{1}{\tau} - \frac{1}{\tau_s}\right)^2 + \epsilon_3$$

or

$$\frac{v_r}{v_x} = \frac{\tau_s{}^2}{\tau} + \frac{\tau_s{}^5}{1 + \tau_s{}^2}\left(\frac{1}{\tau} - \frac{1}{\tau_s}\right)^2 + \epsilon_3 \tag{9}$$

where ϵ_3 is of the order of the third power of the argument; that is,

$$\lim_{\tau \to \tau_s} \frac{\epsilon_3}{\left(\frac{1}{\tau} - \frac{1}{\tau_s}\right)^2} = 0$$

Equation (9) can be written as

$$\frac{v_r}{v_x} = \frac{\tau_s{}^2}{\tau}\left[1 + \frac{\tau_s/\tau}{1 + \tau_s{}^2}(\tau - \tau_s)^2 + \bar{\epsilon}_3\right] \tag{9a}$$

where $\bar{\epsilon}_3$ is also of third order. In cases in which the difference $\tau - \tau_s$ remains small throughout the flow field behind the shock wave (that is, where the shock wave lies close to the cone surface), equation (9a) can be approximated by

$$\frac{v_r}{v_x} = \frac{\tau_s{}^2}{\tau} \tag{10}$$

This relation constitutes the basis of the approximate solution to be developed. Of secondary importance in their effects upon the accuracy of the approximation expressed in equation (10) are (a) the tendency, most pronounced in the case of a cone of small vertex angle, of the factor τ_s/τ partially to offset the increase of error with increasing τ and (b) the slight tendency of the factor $1/\left(1 + \tau_s{}^2\right)$ to improve the accuracy with increasing cone angle. The effect of γ enters only in terms of higher order. (See, for example, ref. 3, p. 466.)

The results of the foregoing analysis may be stated as follows: In an angular neighborhood of the body-cone surface, the variation of the velocity ratio v_r/v_x with the variable $\tau \equiv \tan\theta$ is, to a first approximation, given by equation (10).

Consider now an axially symmetric cylindrical coordinate system with origin at the vertex of the body cone such that the x-axis coincides with the cone axis and has its positive direction downstream, and let r denote the distance of the point (x,r) in this system from the axis. Then,

$$\frac{r}{x} = \tau \tag{11}$$

In this coordinate system, the differential equation of a streamline is

$$\frac{dr}{dx} = \frac{v_r}{v_x}$$

With the use of equation (11) and the approximate equation (10), the differential equation of an arbitrary streamline becomes

$$\frac{dr}{dx} = {\tau_s}^2 \frac{x}{r}$$

Separating the variables and integrating gives, as the equation of the family of streamlines,

$$r^2 = {\tau_s}^2 x^2 + \text{Constant} \tag{12}$$

When the arbitrary constant of integration is permitted to take on all positive values, equation (12) represents a family of geometrically similar hyperbolas with centers at the origin (the cone vertex), transverse axes through the origin and normal to the cone axis, and common eccentricity $\csc\theta_s$. The hyperbolas of the family have common asymptotes which are generators of the body-cone surface and whose equations are

$$r = \pm\tau_s x$$

For the set of positive values of the integration constant, this family of curves completely fills the portion of space exterior to the solid body cone and its natural counterpart which is obtained by extending the cone generators through the vertex. The conjugate hyperbolas, obtained by assigning negative values to the arbitrary constant in equation (12), lie inside the cone and are without physical significance in this case. The portions of these hyperbolas upstream of the shock wave are discarded and replaced by parallel flow in the subsequent formulation of the approximate solution for the flow over cones.

In the preceeding analysis, it has been shown that in an angular neighborhood of the cone surface the streamlines are, to a first approximation, portions of geometrically similar hyperbolas whose common center is the cone vertex and whose asymptotes are extended generators of the cone surface.

The geometric form of the streamlines near the surface depends, to a first approximation, solely on the form of the surface (that is, the vertex angle of the cone), but is independent of the velocity at the surface and, therefore, independent of the free-stream Mach number. It has been pointed out to the author that equation (12) can also be obtained by applying the one-dimensional continuity equation to a stream tube adjacent to the surface of the body cone. In this connection it is noted that, to a first approximation, the resultant speed $\sqrt{u^2 + v^2} \equiv \sqrt{{v_x}^2 + {v_r}^2}$ is constant near the body-cone surface; since the flow is isentropic, a similar statement applies to the density. However, it should be noted, also, that constant resultant speed and constant density are not a part of the approximate solution to be developed.

DERIVATION OF APPROXIMATE SOLUTION

In order to develop an approximate solution for axially symmetric flow over a cone with an attached shock wave, equation (10) is now assumed to hold throughout the flow field behind the shock wave. If the coefficient τ in the first of equations (7), namely,

$$\frac{dv_x}{d\tau} = -\tau \frac{dv_r}{d\tau}$$

is replaced by its value as determined by equation (10), the resulting differential equation can be written in the form

$$\frac{d}{d\tau}\left(\log_e v_x + {\tau_s}^2 \log_e v_r\right) = 0$$

Hence,

$$v_x\left(v_r\right)^{\tau_s^2} = \text{Constant}$$

Thus,

$$v_x\left(v_r\right)^{\tau_s^2} = v_{x,s}\left(v_{r,s}\right)^{\tau_s^2} \tag{13}$$

When the approximate equations (10) and (13) are solved simultaneously with the use of equation (8), the resulting approximate equations for the velocity components v_x and v_r can be written as

$$\frac{v_x}{v_{x,s}} = \left(\frac{\tan\,\theta}{\tan\,\theta_s}\right)^{\sin^2\theta_s} \tag{14a}$$

$$\frac{v_r}{v_{r,s}} = \left(\frac{\tan\,\theta_s}{\tan\,\theta}\right)^{\cos^2\theta_s} \tag{14b}$$

Since the boundary values of the velocity components are related to the resultant velocity u_s at the body-cone surface by the equations

$$v_{x,s} = u_s \cos\,\theta_s \tag{15a}$$

$$v_{r,s} = u_s \sin\,\theta_s \tag{15b}$$

it is seen that equations (14) essentially give the velocity components in terms of the variable θ and any two of the four parameters $v_{x,s}$, $v_{r,s}$, u_s, and θ_s. (In the solutions of refs. 2, 3, and 4, the parameters which have been chosen are u_s and θ_s.)

In deriving the approximate relations (14), use was made of the first of equations (7), but not the second. The significance of neglecting the second of equations (7) is discussed in appendix A.

The approximate equations (14) relating the velocity field behind the shock wave to the velocity at the body-cone surface may, in some cases, be combined with information on strong shock waves in a real gas to obtain estimates of the effects of vibrational excitation and dissociation on the flow over conical tips in the hypersonic range. Such estimates will be most accurate when the relaxation distance measured along a streamline is everywhere negligibly small in comparison with the length of the conical tip measured along a generator, that is, when the distortion of the conical form of the flow field caused by relaxation effects is insignificant.

In returning to the development of the theory based upon the approximate equation (10), consideration is again restricted to perfect gases; thus, in particular, the change in state of the gas across the shock wave is assumed to be of the Rankine-Hugoniot type and the effects of vibrational excitation and dissociation are assumed not to occur. The theory of the oblique shock wave gives, for the velocity components $v_{x,w}$ and $v_{r,w}$ immediately behind the wave, the exact equations

$$\left.\begin{aligned} v_{x,w} &= \left(\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}\right)^{1/2} \left[1 - \frac{2}{\gamma + 1}\left(\sin^2\theta_w - \frac{1}{M^2}\right)\right] \\ v_{r,w} &= \left(\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}\right)^{1/2} (\cot \theta_w) \frac{2}{\gamma + 1}\left(\sin^2\theta_w - \frac{1}{M^2}\right) \end{aligned}\right\} \tag{16}$$

In line with the previous notation, the velocity components $v_{x,w}$ and $v_{r,w}$ are parallel and perpendicular, respectively, to the free-stream direction; as before, they are referred to the limiting velocity c. The free-stream Mach number and the shock-wave angle are denoted by M and θ_w, respectively.

Since equation (10) is assumed to hold throughout the flow field behind the shock wave, immediately behind the wave it becomes

$$\frac{v_{x,w}}{v_{r,w} \tan \theta_w} = \frac{1}{\tau_s{}^2} = \frac{1}{\sin^2\theta_s} - 1$$

When the velocity components in this equation are replaced by their values as given in equations (16), the resulting approximate equation is

$$\frac{2}{\gamma + 1}\left(\sin^2\theta_w - \frac{1}{M^2}\right) = \sin^2\theta_s$$

or

$$\sin^2\theta_w = \frac{1}{M^2} + \frac{\gamma + 1}{2}\sin^2\theta_s \tag{17}$$

In this approximate equation the shock-wave angle, the free-stream Mach number, the body-cone angle, and the ratio of specific heats are related in such a way that the equation may be solved for any one of the four parameters in terms of the other three. It is evident that the equation becomes exact in the limit as the body-cone angle approaches zero.

An interesting theoretical comparison between the cone and the wedge can be obtained by using the approximate equation (17) and its exact counterpart from the oblique-shock-wave theory (ref. 11, p. 57). For infinite Mach number, it follows from equation (17) that the limit of the ratio of the shock-wave angle to the cone angle, as the cone angle approaches zero, is given by

$$\lim_{\theta_s \to 0}\left(\frac{\theta_w}{\theta_s}\right)_{M=\infty} = \sqrt{\frac{\gamma + 1}{2}}$$

The corresponding limit in the case of the wedge is easily found to be $(\gamma + 1)/2$.

The approximate equation (17), for given values of the free-stream Mach number M and the cone angle θ_s, gives essentially one significant value for the shock-wave angle θ_w. Moreover, this value is, in general, a reasonably good approximation to the physically observed or "first solution" value of θ_w, but not to the larger "second solution" value. (See ref. 4.) This is consistent with the limitation of the applicability of equation (10) to cases in which the shock wave lies near the cone surface.

The approximate expressions that result for the velocity components immediately behind the shock wave, when equation (17) is used to replace θ_w in equations (16), are

$$v_{x,w} = \left(\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}\right)^{1/2} \cos^2\theta_s \tag{18a}$$

$$v_{r,w} = \left(\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}\right)^{1/2} \left[\frac{M^2 - \left(1 + \frac{\gamma + 1}{2} M^2 \sin^2\theta_s\right)}{1 + \frac{\gamma + 1}{2} M^2 \sin^2\theta_s}\right]^{1/2} \sin^2\theta_s \tag{18b}$$

Equations (18), then, give the approximate values of the velocity components immediately behind the wave in terms of the cone angle and the free-stream Mach number.

Using equation (14a), evaluated just behind the shock wave, together with equation (15a) yields the approximate relation

$$u_s = \frac{v_{x,w}}{\cos\theta_s} \left(\frac{\tan\theta_s}{\tan\theta_w}\right)^{\sin^2\theta_s} \tag{19}$$

When equations (17) and (18a) are used to eliminate θ_w and $v_{x,w}$ from equation (19), the resulting expression for u_s is

$$u_s = \left(\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}\right)^{1/2} (\cos\theta_s) \left[\frac{M^2 - \left(1 + \frac{\gamma + 1}{2} M^2 \sin^2\theta_s\right)}{1 + \frac{\gamma + 1}{2} M^2 \sin^2\theta_s} \tan^2\theta_s\right]^{\frac{\sin^2\theta_s}{2}} \tag{20}$$

In this approximate equation for the velocity at the surface of the body cone, as in the case of equations (18), the parameters are the free-stream Mach number M and the cone (semivertex) angle θ_s.

Thus, equations (14), (15), (17), (18), and (20) give an approximate description of the velocity field and the orientation of the attached shock wave in terms of the cone angle, the free-steam Mach number, and the adiabatic constant of the gas.

The pressure ratio across the shock wave is given by the exact equation (ref. 11, p. 57)

$$\frac{p_w}{p_1} = \frac{2\gamma}{\gamma + 1} M^2 \sin^2\theta_w - \frac{\gamma - 1}{\gamma + 1}$$

where p_1 is the static pressure in the free stream ahead of the wave and p_w is the static pressure immediately behind the wave. Replacing $\sin^2\theta_w$ in this equation by its approximate value as given in equation (17) gives the approximate expression

$$\frac{p_w}{p_1} = 1 + \gamma M^2 \sin^2\theta_s \qquad (21)$$

Since the flow from immediately behind the shock wave to the surface of the body is assumed to be isentropic,

$$\frac{p_s}{p_w} = \left[\frac{1 - u_s{}^2}{1 - (v_{x,w})^2 - (v_{r,w})^2}\right]^{\frac{\gamma}{\gamma-1}} \qquad (22)$$

where p_s is the static pressure at the surface.

Consequently, the surface pressure coefficient $C_{p,s}$, which is given by

$$C_{p,s} = \frac{2}{\gamma M^2}\left(\frac{p_s}{p_w}\frac{p_w}{p_1} - 1\right) \qquad (23)$$

can be calculated approximately, with the use of equations (18), (20), (21), and (22), for given values of the body-cone angle, the free-stream Mach number, and the ratio of specific heats.

Although it is apparently not worthwhile in the general case to reduce this set of equations (that is, equations (18) and (20) to (23)) to a single expression for the pressure coefficient, considerable simplification does result when the free-stream Mach number becomes infinite. This special case, which is of some theoretical interest, is discussed in appendix B.

COMPARISON WITH OTHER SOLUTIONS

In order to compare the present approximate solution with the three approximate solutions of Eggers and Savin (ref. 8), computations have been carried out for air with $\gamma = 1.405$ in a number of cases which represent rather wide ranges of the cone angle and the free-stream Mach number; the cases have been selected so as to correspond to exact solutions of reference 4. The approximate methods were used to calculate the shock-wave angle θ_w, the surface velocity u_s, and the surface pressure coefficient $C_{p,s}$.

The first approximate solution of Eggers and Savin for the cone is given in the form of equations (12) and (19) of reference 8. The first of these equations is an approximate expression for the flow-direction angle. The Mach number parameter occurring in the equation is evaluated by solving the equation simultaneously with the appropriate oblique-shock-wave equations; a semigraphical method, or its equivalent, may be used to do this. When the necessary parameters have been determined, the two equations give the flow direction and speed, respectively, in the flow field behind the shock wave as a function of the position angle θ. The solution is primarily applicable to slender cones.

Equations (27) and (30) of reference 8 make up the second approximate solution presented by Eggers and Savin. The computational details are similar to those of the first method. The second solution is primarily applicable to cases in which the shock wave lies close to the body-cone surface and, in this respect, the solution is similar to the present approximation. Reference 8 indicates that the second method, when it is applicable, is to be preferred to the first method.

The third approximate method given by Eggers and Savin (appendix A of ref. 8) corresponds to the limiting case of the solution of the present report for small cone and shock-wave angles.

In a note on the relation of the hypersonic similarity law to axially symmetric flow over cones, Lees (ref. 12) has developed an approximate solution which is comparable to the third method of Eggers and Savin but which is, in general, less uniformly accurate in the calculation of both the shock-wave angle and the surface pressure coefficient. The approximate expression obtained by Lees relating the wave angle to the cone angle, the free-stream Mach number, and the specific-heats ratio can be put into the following form:

$$\theta_w^{\,2} = \frac{1}{M^2} + \frac{\gamma + 1}{2}\,\theta_s^{\,2}\left[1 - \left(\frac{\theta_w}{\theta_s} - 1\right)^2\right]$$

which permits an easy comparison with equation (A7) of reference 8, that is

$$\theta_w^2 = \frac{1}{M^2} + \frac{\gamma + 1}{2} \theta_s^2$$

and with equation (17) of the present paper.

The comparison of the three approximate solutions of reference 8 with the approximate solution of the present paper is given in figure 2. In order to facilitate the comparison of the accuracies of the methods, the approximate results are presented in the form of ratios of the calculated values to the exact values as given in reference 4.

Figure 2 generally shows that, for $\theta_s = 5^\circ$, the present method is considerably less accurate than the lengthier first and second methods of reference 8 but is approximately equivalent in accuracy to the third method of reference 8. For $\theta_s = 20^\circ$, the present method is slightly less accurate than the first and second methods of reference 8, but is considerably more accurate than the third method. For $\theta_s = 35^\circ$, the present method is slightly more accurate than the second method of reference 8, and is considerably more accurate than the third method.

Figure 3 shows that the present approximation compares favorably in both accuracy and uniformity of accuracy with the first- and second-order or small-perturbation solutions.

For given values of the cone angle θ_s and the ratio of specific heats γ, equation (20) may be used with successive approximations to determine the free-stream Mach number which corresponds to $u_s = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$, that is, to a Mach number of unity on the surface of the body cone. These calculations have been carried out with $\gamma = 1.405$ for cones ranging in semivertex angle from 0° to 50°, and the results are presented in figure 4 together with the corresponding exact data (ref. 4). The two curves, approximate and exact, give, as a function of the cone angle, the minimum free-stream Mach number for which the flow is entirely supersonic.

A detailed examination of the accuracy of the present approximate solution in calculating the shock-wave angle θ_w, the surface velocity u_s, and the surface pressure coefficient $C_{p,s}$ is given in figure 5 for $\gamma = 1.405$. The computations were performed for several cones with θ_s ranging from 5° to 50° for Mach numbers corresponding to those for which exact calculations were available in reference 4. The lower Mach number limits of the curves correspond to a surface Mach number of unity (consistent with the approximate curve of fig. 4). The tendency toward increased accuracy at the high Mach numbers is evident in all the cases shown in figure 5.

For the limiting case of infinite Mach number, which is discussed in appendix B, the approximate method has been applied to the calculation of the surface pressure coefficient taken, for convenience, in ratio to the Newtonian pressure coefficient $(C_{p,s})_N = 2 \sin^2\theta_s$. The computations were carried through for three values of the ratio of specific heats $\left(\gamma = 1\frac{1}{3}, 1.405, 1\frac{2}{3}\right)$ with θ_s ranging from 0° to 50°. The results are compared in figure 6 with the exact results given in reference 4 for $\gamma = 1\frac{1}{3}$ and $\gamma = 1.405$, and with two points obtained by the numerical integration of equations (7) for $\gamma = 1\frac{2}{3}$.

CONCLUDING REMARKS

It has been shown that the streamlines in an angular neighborhood of the surface of an unyawed circular cone with an attached shock wave are, to a first approximation, portions of hyperbolas. This fact has been used as a basis for the development of an approximate solution in which the shock-wave orientation and the flow field behind the shock wave are given explicitly in terms of the free-stream Mach number, the vertex angle of the body cone, and the ratio of specific heats of the gas. The approximate solution has been compared with other approximate solutions for the cone.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 1, 1955.

APPENDIX A

SOME MATHEMATICAL ASPECTS OF THE APPROXIMATION

It has been shown that equations (14) constitute an approximate solution of the nonlinear system of differential equations (7). Moreover, the first of equations (7) is satisfied identically by the approximate expressions for the velocity components v_x and v_r given in equations (14). Direct calculation from equations (14) and (15) yields the equation

$$\frac{dv_x}{d\tau} = \frac{v_r}{1 + {\tau_s}^2}$$

which is identically satisfied by the approximate expressions for v_x and v_r. Therefore, the approximate expressions for v_x and v_r make up an exact solution of the following system of differential equations:

$$\frac{dv_x}{d\tau} = -\tau \frac{dv_r}{d\tau} = \frac{a^2 v_r}{a^2\left(1 + {\tau_s}^2\right)} \tag{A1}$$

The factor a^2 has been inserted in the numerator and denominator of the right-hand expression in order to permit a better comparison with equations (7). It is now evident that the approximation embodied in equation (10), which led to the approximate solution (14), is equivalent to approximating the system of equations (7) by setting the factors $1 + \tau^2$ and $v_r - \tau v_x$ equal to constants, namely their values at the body-cone surface, and solving the resulting linear system of differential equations exactly.

It is noted from equation (3b) that the factor $v_r - \tau v_x$ can be expressed as

$$v_r - \tau v_x = v \sec \theta = v\sqrt{1 + \tau^2}$$

Thus, setting the factor $v_r - \tau v_x$ in the nonlinear expression of equations (7) equal to its value at the surface of the body cone (that is, zero) is equivalent to neglecting the normal velocity component v in comparison with the local speed of sound a. If this approximation alone is made in the system of equations (1), which is equivalent to the system of equations (7), the resulting system can be written in the form of a second-order, linear differential equation, that is,

$$\frac{d^2u}{d\theta^2} + \cot \theta \frac{du}{d\theta} + 2u = 0 \tag{A2}$$

When this equation is expressed in terms of $\cos \theta$ as the independent variable, it is easily recognized as the Legendre equation for $n = 1$. Its general solution can be written in closed form and is (ref. 13)

$$u = A \cos \theta + B\left(\frac{1}{2} \cos \theta \log_e \frac{1 + \cos \theta}{1 - \cos \theta} - 1\right)$$

where A and B are arbitrary constants to be evaluated in terms of the boundary values. This approximation, which gives good results when $v^2 \ll a^2$, was recognized as a good approximation in the hypersonic speed range a number of years ago by several persons, independently, at the Langley Aeronautical Laboratory. The approximation associated with the Legendre equations has been found to lead to results which are more complex than those which have their origin in the approximate equation (10).

The only approximation which has been introduced in developing the approximate solution of the present report for the axially symmetric flow over a cone with an attached shock wave is that of equation (10). This basic approximation has been found to be equivalent to specifying a flow pattern in which the streamlines are portions of hyperbolas of the type described earlier. Inasmuch as the second of differential equations (7) was not used in deriving the approximate solution, the problem was not overdetermined by the introduction of equation (10). The relation of the second of equations (7) to the approximate solution is clear from the form of the approximate system of differential equations (A1) which was found to be equivalent to introducing the approximation embodied in equation (10).

APPENDIX B

THE LIMITING CASE OF INFINITE MACH NUMBER

For the limiting case of infinite free-stream Mach number, equations (18) and (20) to (23) yield the approximate expression

$$\frac{\left(C_{p,s}\right)_{M=\infty}}{\left(C_{p,s}\right)_{N}} = \left[\frac{\gamma + 1}{2\gamma} \frac{1 - \cos^2\theta_s \left(\dfrac{1 - \dfrac{\gamma + 1}{2} \sin^2\theta_s}{\dfrac{\gamma + 1}{2} \cos^2\theta_s}\right)^{\sin^2\theta_s}}{\sin^2\theta_s}\right]^{\frac{\gamma}{\gamma-1}} \tag{B1}$$

where $\left(C_{p,s}\right)_N$ denotes the surface pressure coefficient according to the inelastic-impact theory of Newton, that is,

$$\left(C_{p,s}\right)_N = 2 \sin^2\theta_s$$

When $\sin^2\theta_s \ll 1$, the application of equation (B1) is, under normal observances of accuracy, subject to rather large computational errors. For such cases, it is preferable to use an approximate form of equation (B1) obtained by expanding the quantity

$$1 - \cos^2\theta_s \left(\frac{1 - \dfrac{\gamma + 1}{2} \sin^2\theta_s}{\dfrac{\gamma + 1}{2} \cos^2\theta_s}\right)^{\sin^2\theta_s}$$

in powers of $\sin^2\theta_s$. The resulting series is

$$\left(1 + \log_e \frac{\gamma + 1}{2}\right) \sin^2\theta_s + \left[\frac{\gamma - 1}{2} - \log_e \frac{\gamma + 1}{2} - \frac{1}{2} \left(\log_e \frac{\gamma + 1}{2}\right)^2\right] \left(\sin^2\theta_s\right)^2 + \dots$$

Consequently, for $\sin^2\theta_s \ll 1$, equation (B1) takes the approximate form

$$\frac{(C_{p,s})_{M=\infty}}{(C_{p,s})_N} = \left(\frac{\gamma + 1}{2\gamma}\left\{1 + \log_e \frac{\gamma + 1}{2} + \left[\frac{\gamma - 1}{2} - \log_e \frac{\gamma + 1}{2} - \frac{1}{2}\left(\log_e \frac{\gamma + 1}{2}\right)^2\right]\sin^2\theta_s\right\}\right)^{\frac{\gamma}{\gamma-1}} \tag{B2}$$

For $\theta_s \leqq 10^\circ$, this equation is generally to be preferred to equation (B1) in performing numerical computations.

Finally, equation (B1) permits one to write

$$\lim_{\gamma \to 1} \log_e \frac{(C_{p,s})_{M=\infty}}{(C_{p,s})_N} = 0$$

after the indeterminate form concerned has been resolved. Thus, it follows that

$$\lim_{\gamma \to 1} (C_{p,s})_{M=\infty} = (C_{p,s})_N \tag{B3}$$

Since the inelastic-impact theory of Newton is applicable to the flow over an unyawed circular cone of a gas with an infinite number of internal degrees of freedom, equation (B3) shows that equation (B1) becomes exact in the limit as $\gamma \to 1$. Similarly, equation (17) predicts correctly that, for infinite Mach number and $\gamma = 1$, the shock wave lies on the body-cone surface. For given values of the velocity and specific enthalpy of the gas ahead of the shock wave, the free-stream Mach number necessarily becomes infinite in the limit as the ratio of specific heats approaches unity. At the same time straightforward analysis more generally shows that the approximate expression (17) for the shock-wave angle and the approximate expression for the surface pressure coefficient which can be obtained from equations (18) and (20) to (23) both become exact in the limit as $\gamma \to 1$; that is, they reduce to the corresponding expressions in the inelastic-impact theory of Newton. Therefore, the present approximate solution for the cone becomes exact in the limit as $\gamma \to 1$ whether the free-stream Mach number is restricted to the value infinity or is not restricted.

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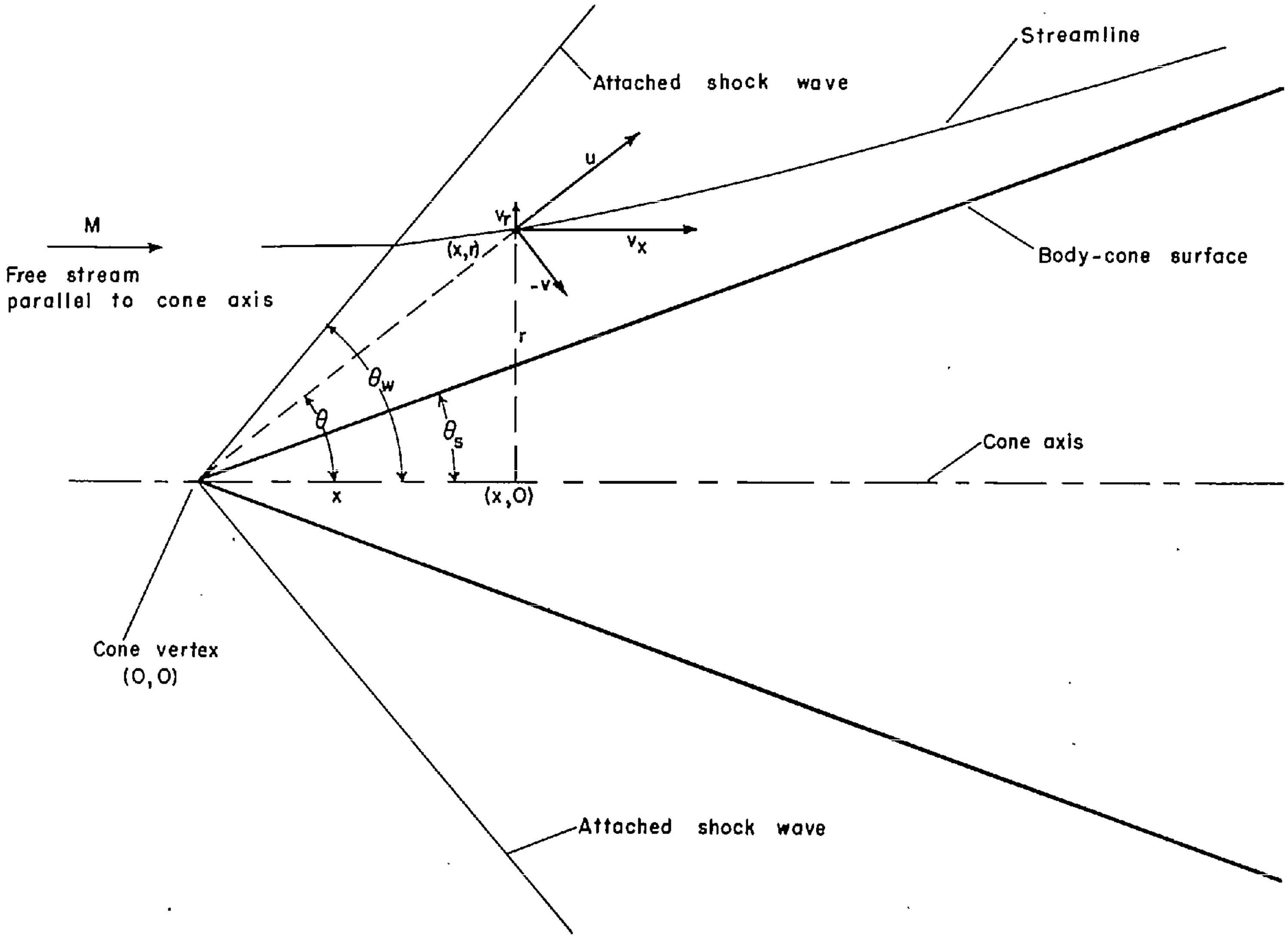
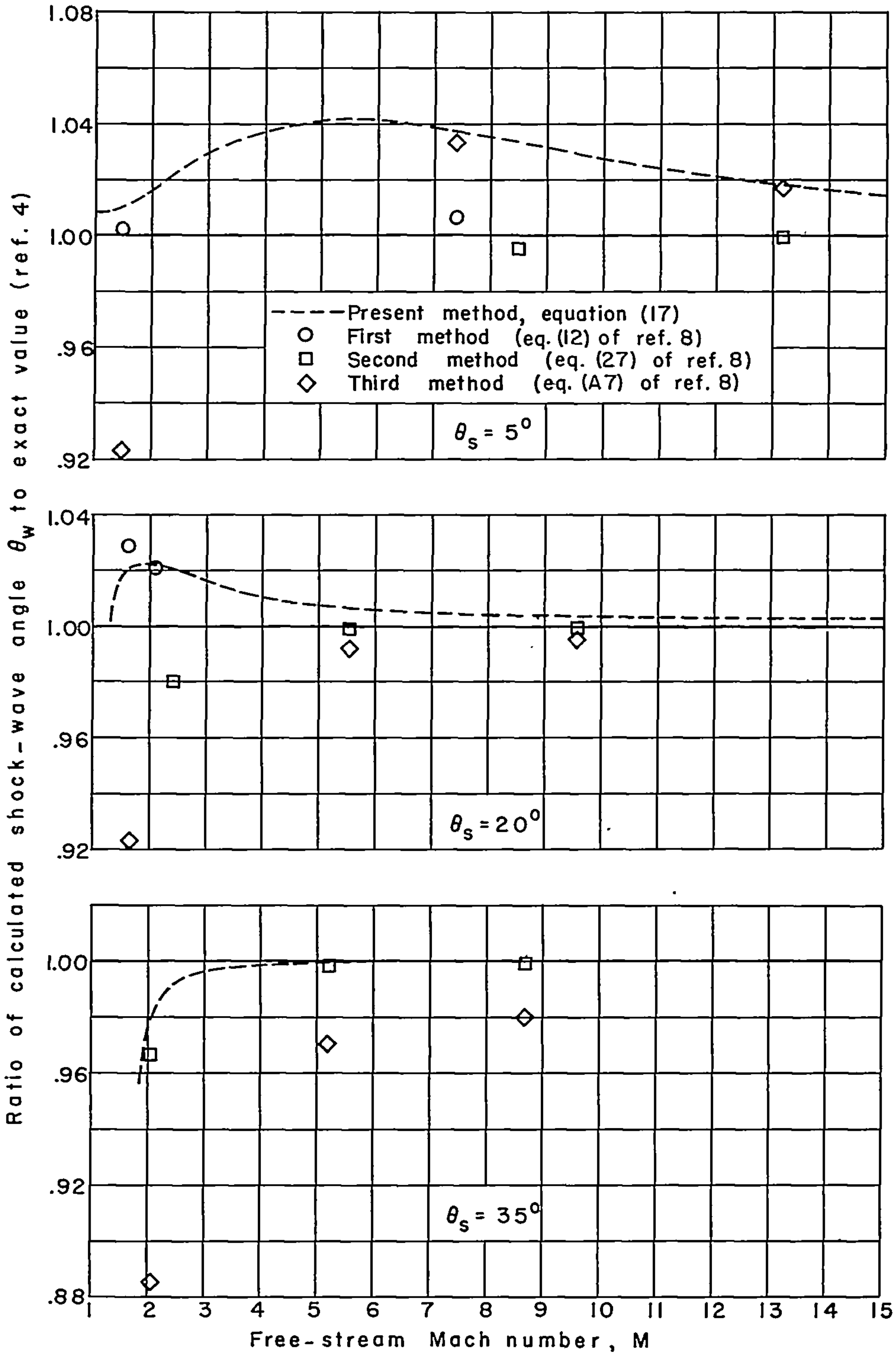
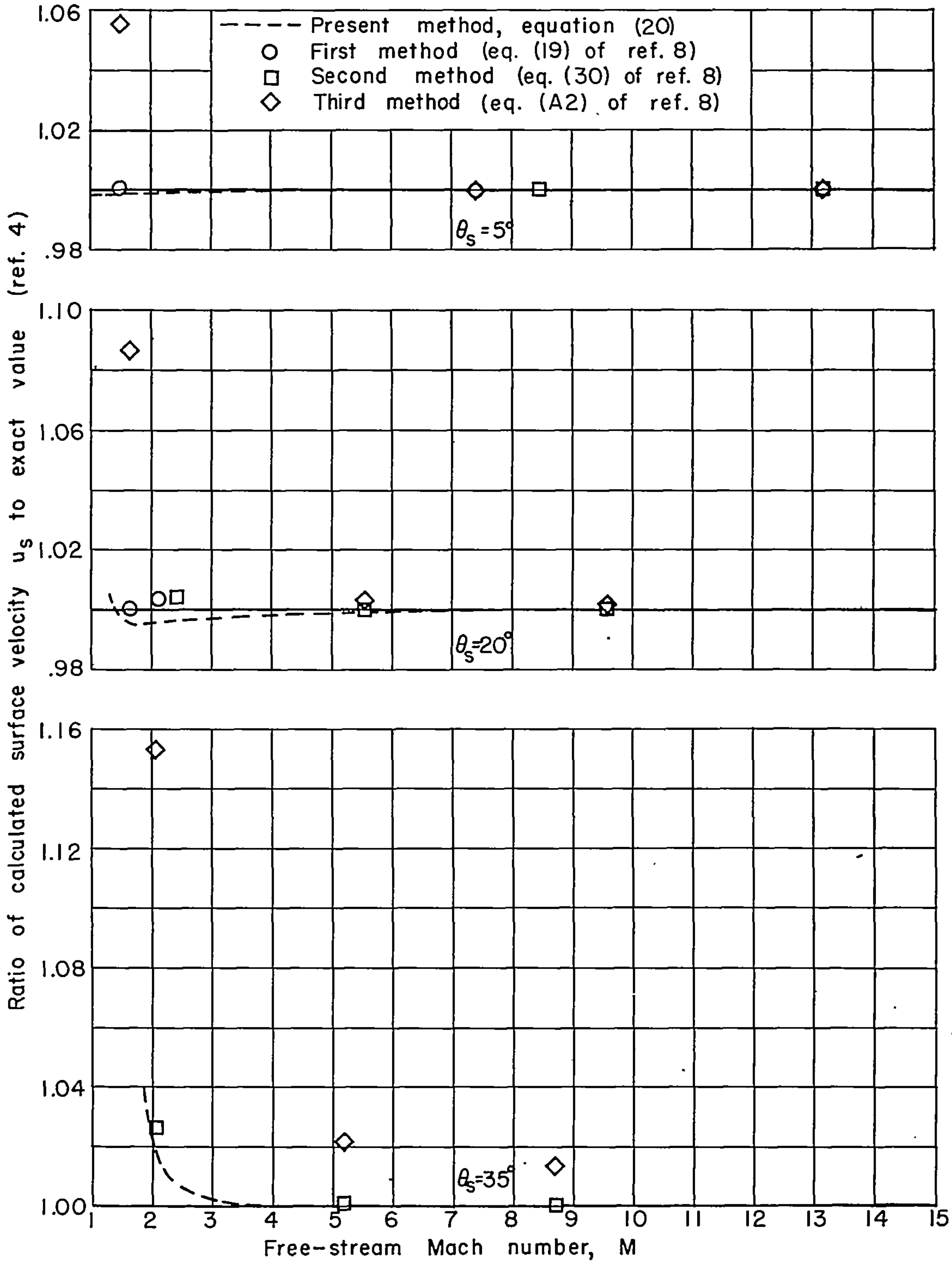


Figure 1.- Axially symmetric flow over cone with attached shock wave.



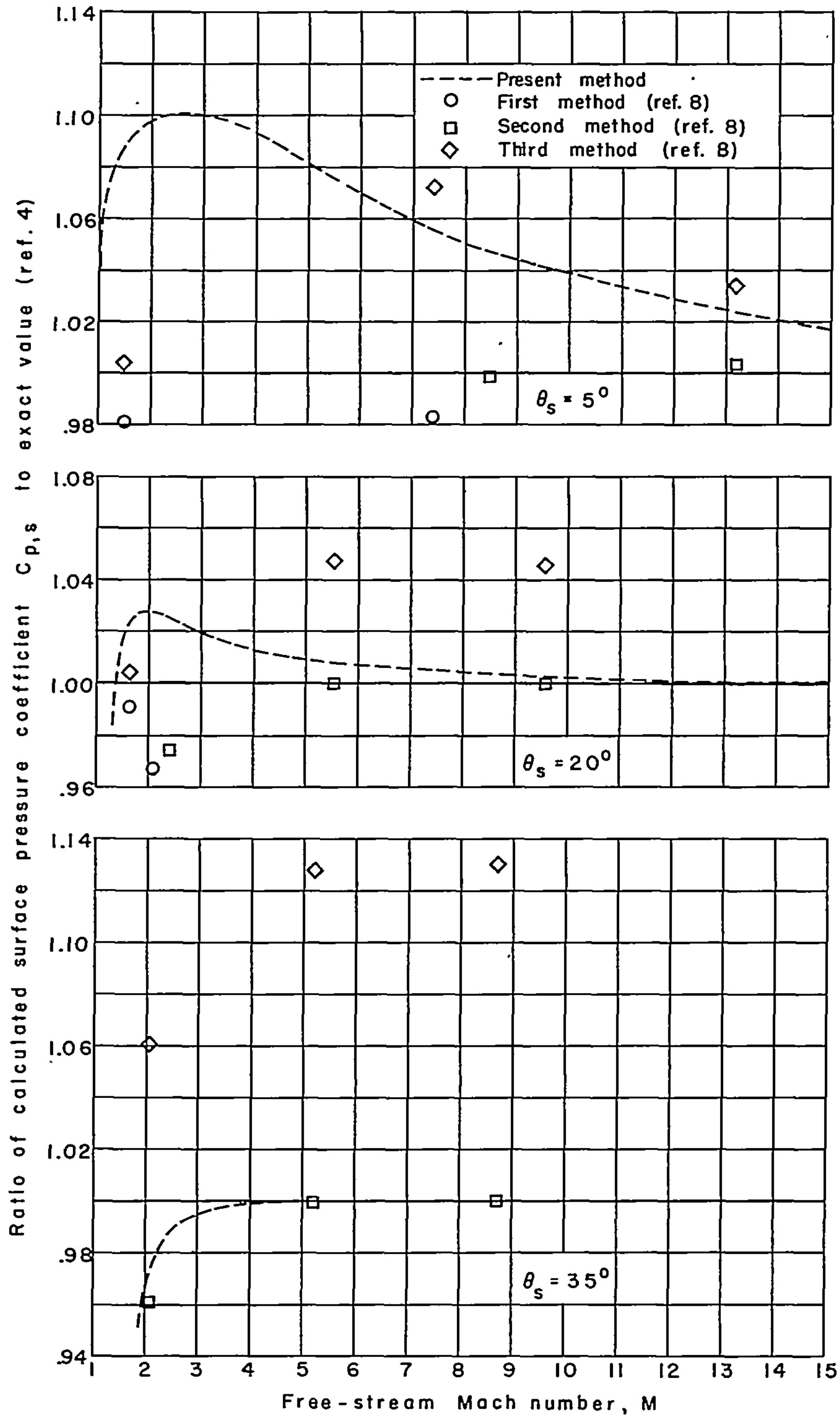
(a) Shock-wave angle.

Figure 2.- Comparison of present method with approximate methods of reference 8. γ = 1.405.



(b) Surface velocity.

Figure 2.- Continued.



(c) Surface pressure coefficient.

Figure 2.- Concluded.

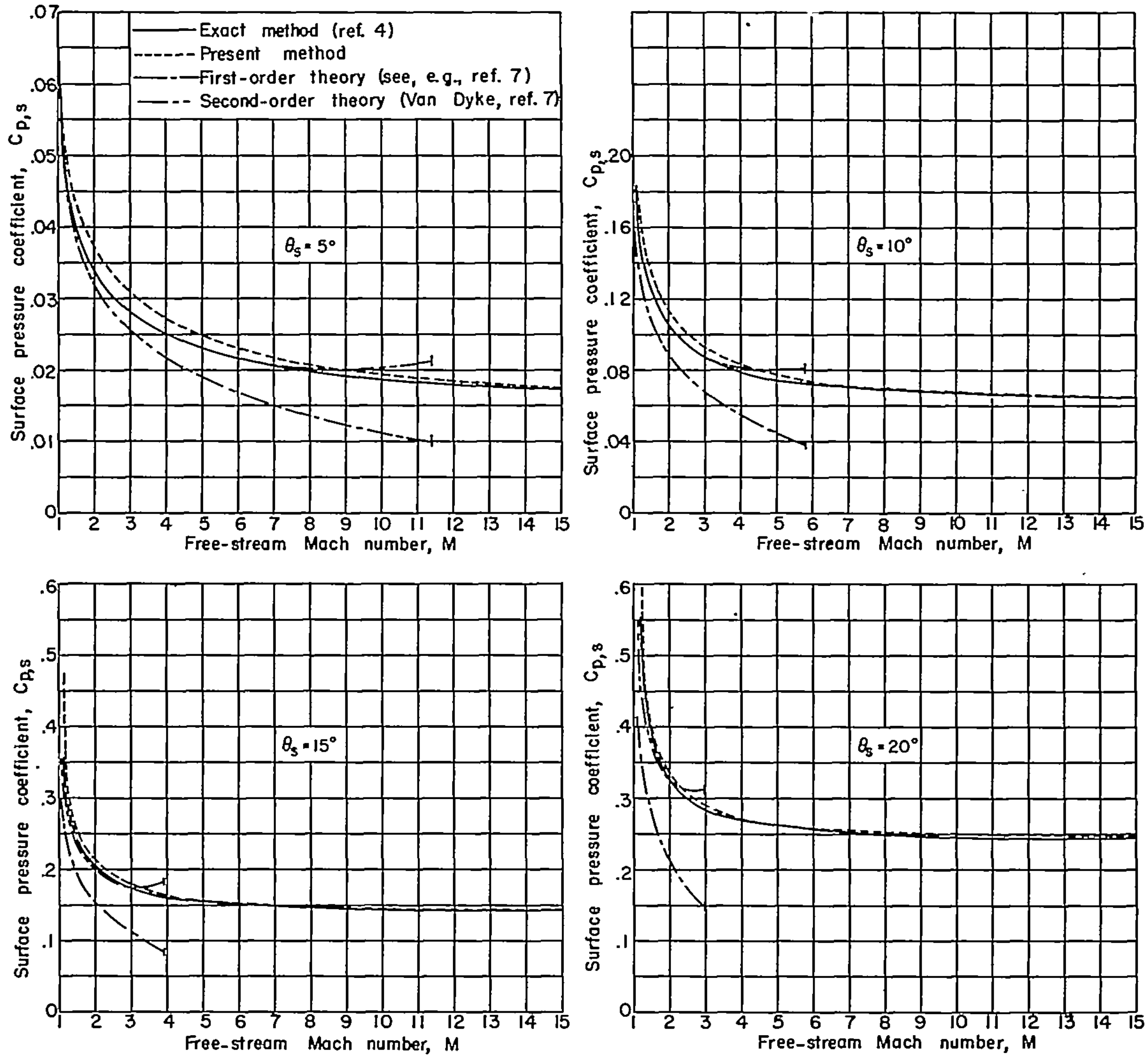


Figure 3.- Surface pressure coefficients for cones of various vertex angles. $\gamma = 1.405$.

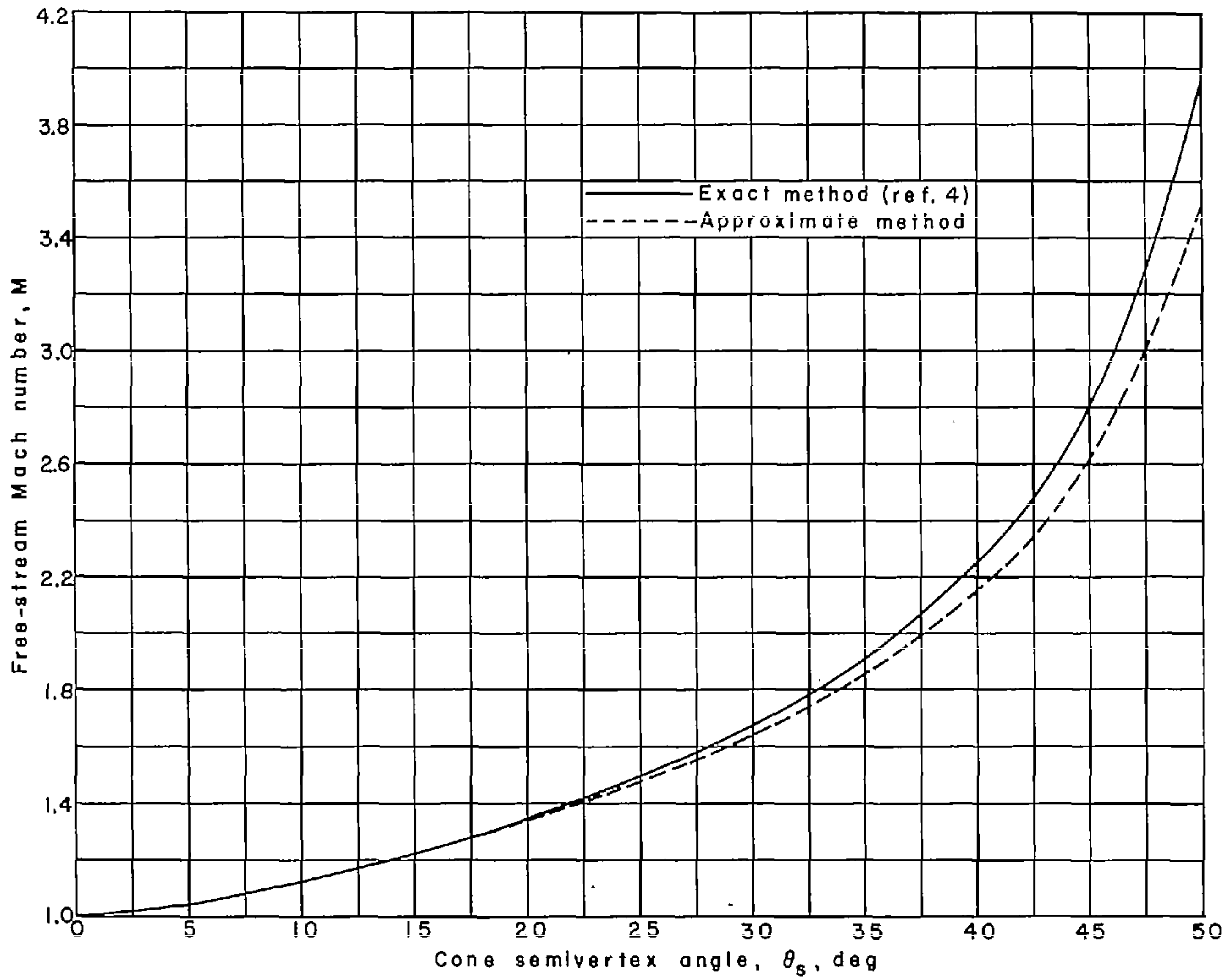
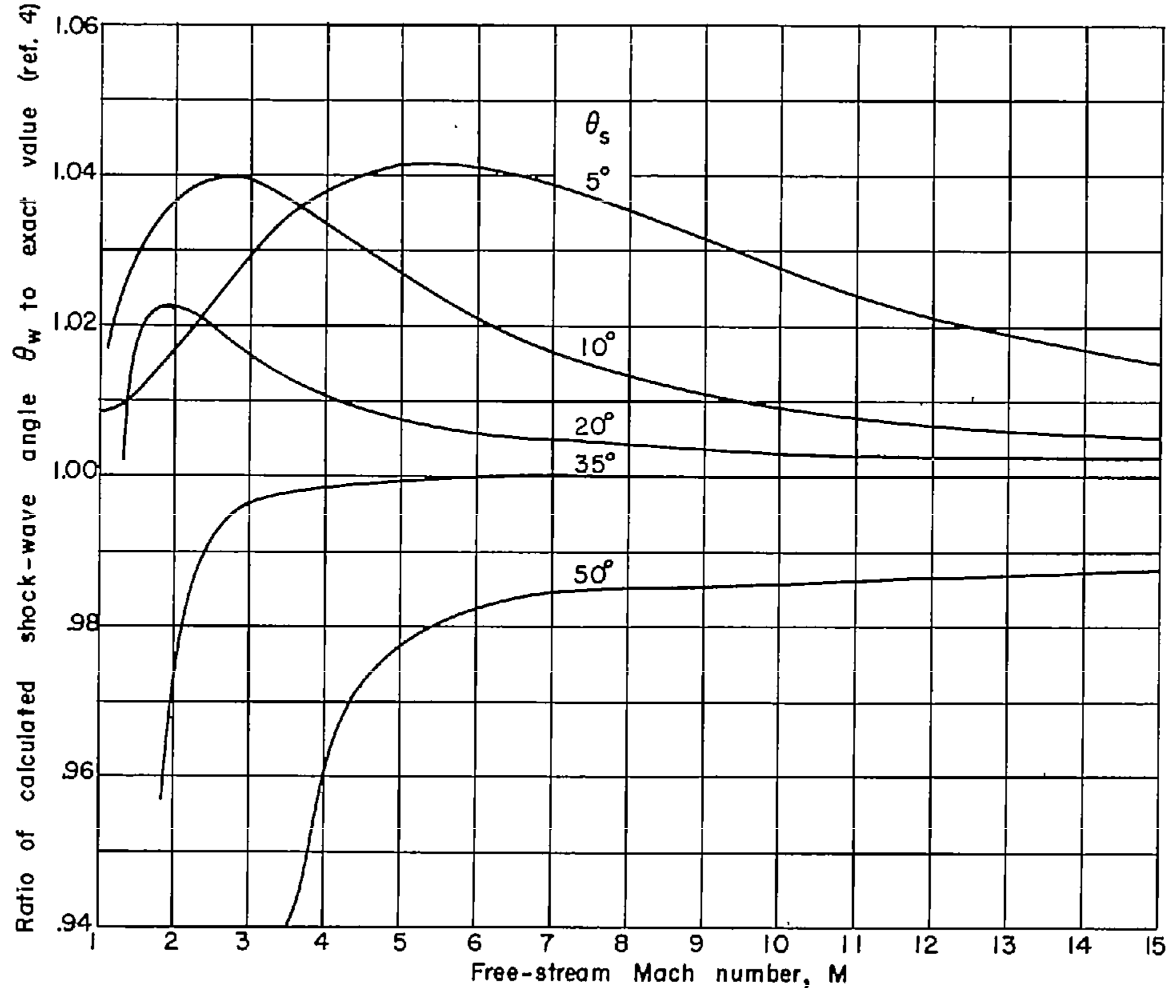
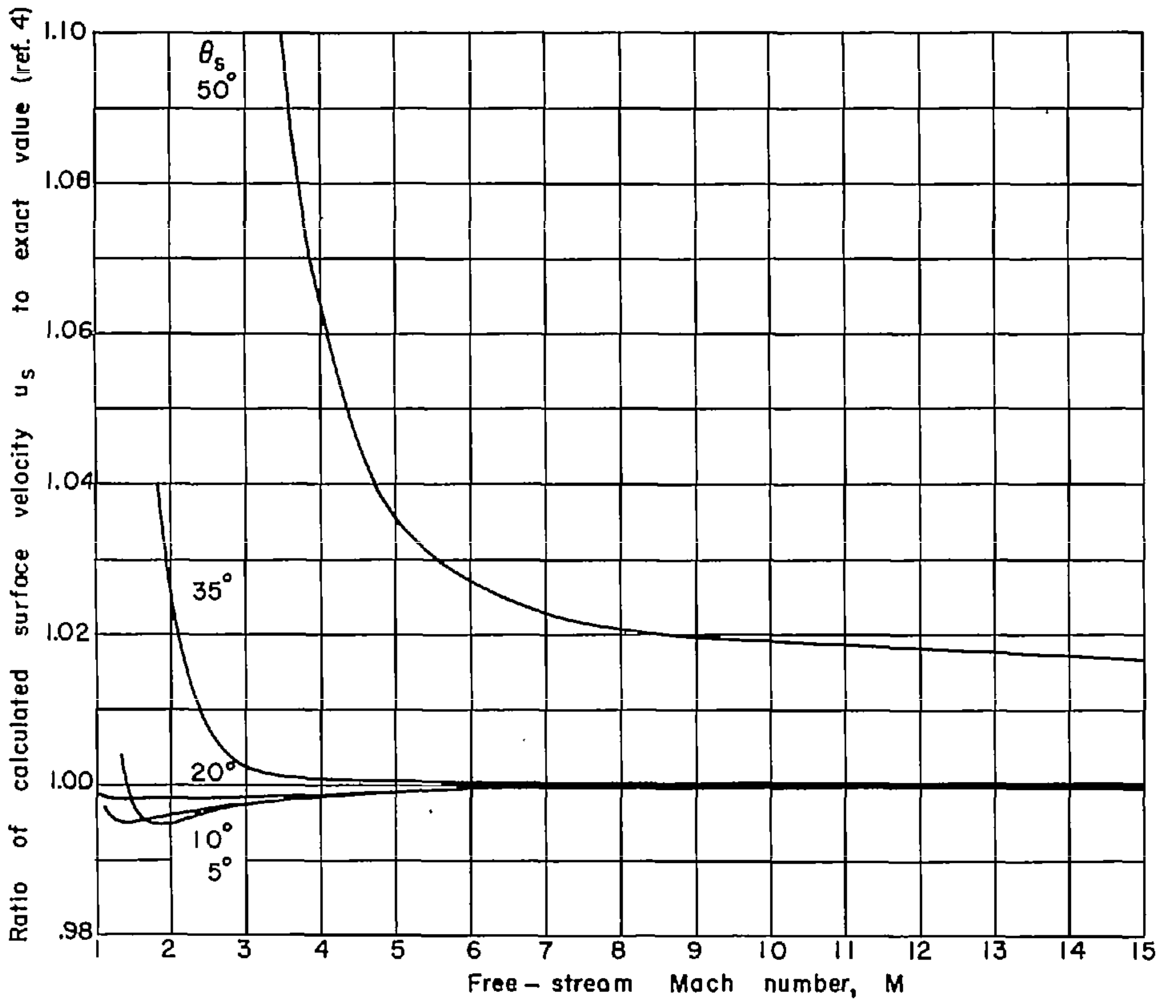


Figure 4.- Free-stream Mach number plotted against cone semivertex angle for surface Mach number of 1.



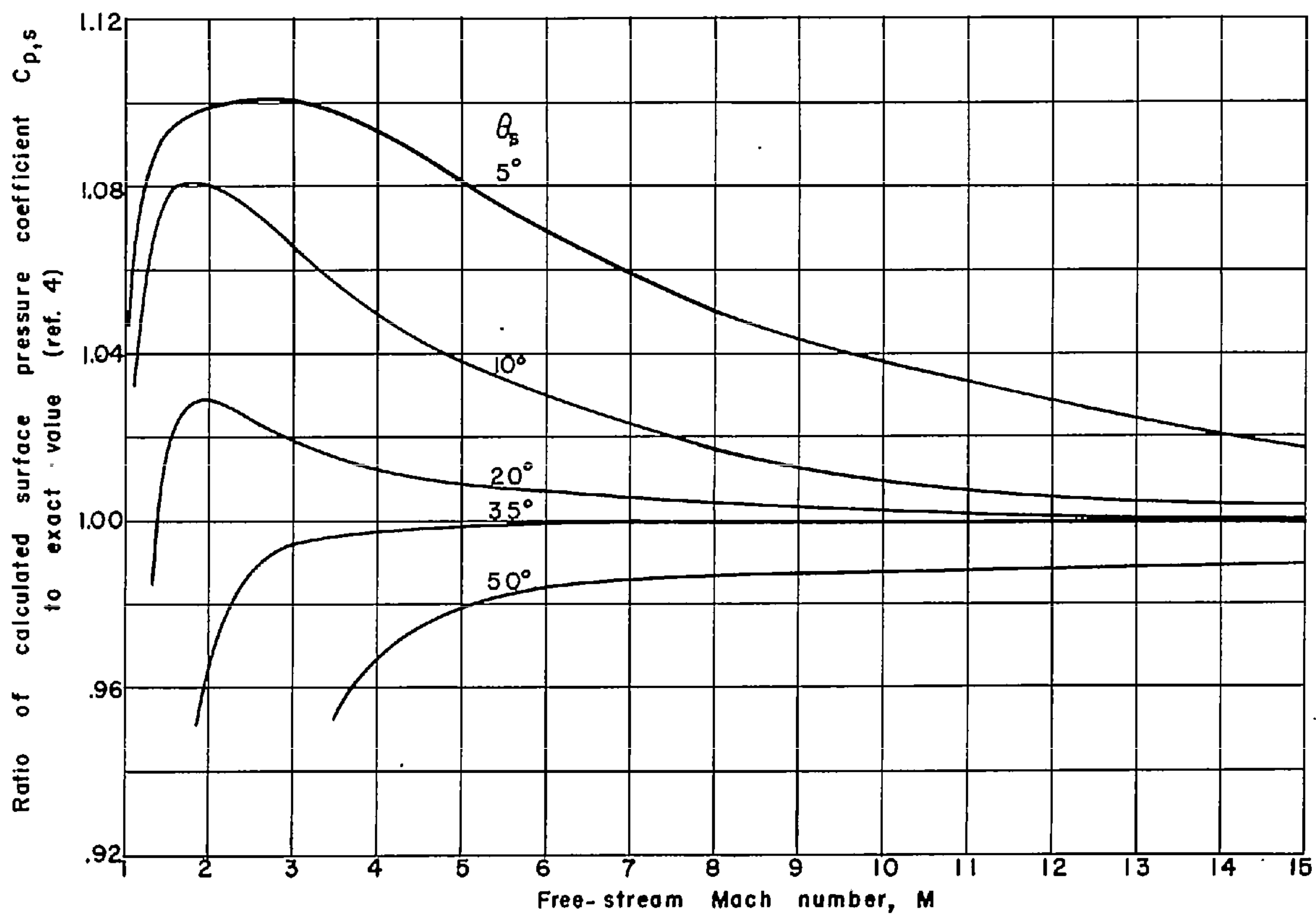
(a) Shock-wave angle.

Figure 5.- Detailed comparison of present method with exact method (ref. 4). $\gamma = 1.405$.



(b) Surface velocity.

Figure 5.- Continued.



(c) Surface pressure coefficient.

Figure 5.- Concluded.

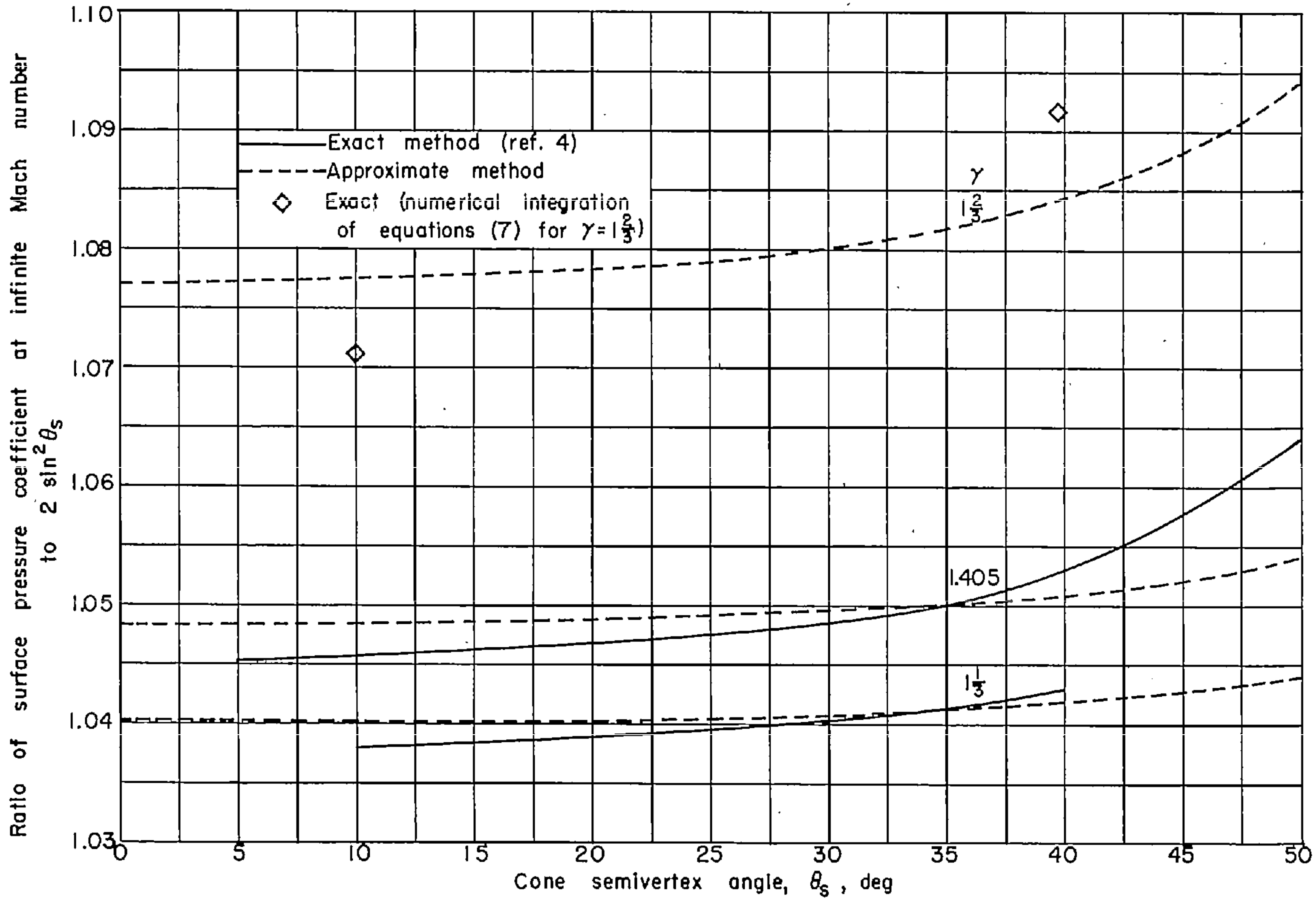


Figure 6.- Comparison of surface pressure coefficients at infinite Mach number.